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Interaction of coupled higher order nonlinear Schrödinger equation solitons

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Abstract. The novel inelastic collision properties of two-soliton interaction for an n-component coupled higher order nonlinear Schrödinger equation are studied. Some interesting features of three soliton interactions, related to the integrability of the n-component coupled higher order nonlinear Schrödinger equation are also discussed.

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1 Introduction

The subject of integrable model is very fascinating largely because of its innumerable symmetries and a special class of solutions known as soliton solutions. Only a few systems in the nature are known to be integrable. The coupled nonlinear Schrödinger equation (CNLS) and its higher order generalization namely coupled higher order nonlinear Schrödinger equation (CHNLS) are some of the examples of integrable equations which have direct relevance in the propagation of optical solitons in Kerr type nonlinear fibre. Coupled integrable systems have many important applications in photorefractive crystals [1] and also in all optical computations [2]. The cross phase modulation (CPM) phenomena in CNLS equation provides an interesting pulse shepherding effect to align the time arrival of the pulses [3]. The CPM phenomena along with group velocity dispersion (GVD) can also be utilized in compressing optical pulses at the one soliton level [4]. Recently the existence of multi-component solitons are experimentally established [5].

In this paper we have considered some interesting features of CHNLS equation, which describes the propagation of the optical pulses of very short width, of the order of 10^{-15} s. We have studied the novel inelastic collision of two solitons for an n-component CHNLS equation. The inelastic collisions among the three solitons are also studied. The three soliton interactions may be interpreted as a combination of two-soliton interactions occurring at three different points. This is in compliance with the three particle interactions in integrable systems. The energy ex-

change among the components of three solitons may occur at any of the three two-soliton interaction points. In this context, it is important to note that the phenomena of shape changing *vis a vis* inelastic collisions and their consequences have been observed for the CNLS equation by considering two soliton solutions of both two and three components and subsequently n-component generalization of two soliton solutions and n-complexes have been studied [2]. Moreover, from integrability point of view study of the collisions of three solitons becomes indispensable and one needs to address this issue separately.

The paper is organized in the following sequence. In Section 2 the CHNLS equation is introduced and its N soliton solution is given explicitly. In Section 3 two-soliton interactions and its asymptotic analysis are obtained showing the novel inelastic collisions among the components of the solitons. Some interesting features of three soliton interactions, related to the integrability of the n-component CHNLS equation are addressed in Section 4. Section 5 is the concluding one.

2 CHNLS equation and N-soliton solution

The n-component CHNLS equation incorporating the effects of Kerr type nonlinearity and stimulated Raman scattering may be written as

$$E_{iz} + iE_{i\tau\tau} + 2i\left(\sum_{j=1}^{n} E_j^* E_j\right) E_i + \varepsilon E_{i\tau\tau\tau}$$

$$+ 3\varepsilon\left(\sum_{j=1}^{n} E_j^* E_j\right) E_{i\tau} + 3\varepsilon\left(\sum_{j=1}^{n} E_j^* E_{j\tau}\right) E_i = 0 \quad (1)$$

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where E_i is n-component electric field. z and τ denote the direction of propagation and time variable respectively. The parameter ε is the ratio of the width of the spectra $\Delta\omega$ to the carrier frequency ω such that $\varepsilon = \frac{\Delta\omega}{\omega} < 1$ and consequently the parameter ε cannot be set to zero for optical solitons.

A particular gauge equivalent form of (1) may be written as [6]

$$q_{ix} + \varepsilon q_{ittt} + 3\varepsilon \sum_{j=1}^{n} \left(q_j^* q_j\right) q_{it} + 3\varepsilon \sum_{j=1}^{n} \left(q_j^* q_{jt}\right) q_i = 0 \quad (2)$$

where the transformation relations are given by

$$E_i(z,\tau) = q_i(x,t) e^{\left(-\frac{1}{27\varepsilon^2}x - \frac{1}{3\varepsilon}t\right)}$$
$$x = z$$
$$t = \tau - \frac{1}{3\epsilon}z. \quad (3)$$

The transformed equation (2) is known as the coupled complex modified KdV equation (CCMKdV) whose N soliton solutions for n-component field is known [6]. Notice that the CCMKdV equation in (2) is convenient particularly for the Lax representation of the system and consequently for applying the inverse scattering method to obtain N-soliton solution. However, as a result of the transformations (3), there is no change of the envelope function, but for a constant shift in velocity for all solitons. Consequently, the shape of the pulses for CCMKdV equation and CHNLS equation remains the same i.e., $|E_i| = |q_i|$ and the transformation will not affect the subsequent results of the two gauge equivalent systems.

The N- soliton solution of the n-component CCMKdV equation may be written in the most compact form as [6],

$$q_i(x,t) = -2 \sum_{j=1}^{N} (\mathbf{B}\mathbf{C}^{-1})_{ij} e^{-i\lambda_j^* t} \quad (4)$$

where, $\mathbf{B}$ and $\mathbf{C}$ are respectively $n \times N$ and $N \times N$ matrices whose explicit forms are given by

$$(\mathbf{B})_{ij} = i\mathcal{C}_{n+1i}^{(j)}(0) e^{-8i\varepsilon\lambda_j^{*3}x - i\lambda_j^* t} \quad (5)$$

and

$$(\mathbf{C})_{ij} = \sum_{p=1}^{n} \sum_{k=1}^{N} \frac{\mathcal{C}_{n+1p}^{(j)}(0)\mathcal{C}_{n+1p}^{*(k)}(0)}{\alpha'_{n+1n+1}(\lambda_j^*)\alpha_{n+1n+1}^{*\prime}(\lambda_k)}$$
$$\times \frac{e^{-i(\lambda_i^* + \lambda_j^* - 2\lambda_k)t + 8i\varepsilon(\lambda_k^3 - \lambda_j^{*3})x}}{(\lambda_k - \lambda_j^*)(\lambda_k - \lambda_i^*)} - \delta_{ij}. \quad (6)$$

In the above equation α_{ij} are the elements of the $(n+1) \times (n+1)$ scattering matrix and λ is the spectral parameter. $\mathcal{C}_{(n+1)p}^{(j)}$ is related to the elements of the scattering matrix at the position of the simple poles, $\mathcal{C}_{(n+1)p}^{(j)} = \alpha_{(n+1)p}(\lambda_j^*)|_{x=0}$ and $\prime$ over α denotes the derivative with respect to λ.

The one soliton solution (1SS) for n-coupled system directly follows from (4, 5, 6) by considering $N = 1$ and is given as

$$q_i(x,t) = \frac{\mathcal{C}_{n+1i}^{(1)} e^{-\frac{R_{11}}{2}} e^{i\eta_{1I}}}{\cosh(\eta_{1R} + \frac{R_{11}}{2})} \quad (7)$$

where η_{1R} and η_{1I} respectively denote the real and imaginary parts of $\eta_1 = -2i\lambda_1^* t - 8i\varepsilon\lambda_1^{*3}x + i\frac{\pi}{2}$ and $e^{R_{11}} = -\frac{\kappa_{11}}{\lambda_{11}^2}$. The complex variables are defined as

$$\kappa_{11} = \sum_{p=1}^{n} \left| \frac{\mathcal{C}_{(n+1)p}^{(1)}}{\alpha'_{n+1n+1}(\lambda_1)} \right|^2$$

and

$$\lambda_{11} = \lambda_1 - \lambda_1^*.$$

For convenience let us define $\lambda_1 = \frac{\lambda_{1R} + i\lambda_{1I}}{2}$, where the subscripts R and I denote the real and imaginary parts and consequently η_{1R} and η_{1I} become

$$\eta_{1I} = -\lambda_{1R}t + \varepsilon\lambda_{1R}(3\lambda_{1I}^2 - \lambda_{1R}^2)x + \frac{\pi}{2} \quad (8)$$

$$\eta_{1R} = -\lambda_{1I}[t - \varepsilon(\lambda_{1I}^2 - 3\lambda_{1R}^2)x]. \quad (9)$$

From the argument of the cosh function of the 1SS (7) and (9), it is straight forward to identify the width of the soliton pulse as $\Gamma = |\lambda_{1I}|^{-1}$ and the soliton travels with a group velocity $V_g = [\epsilon(\lambda_{1I}^2 - 3\lambda_{1R}^2)]^{-1}$ in the positive x direction when $\lambda_{1I}^2 > 3\lambda_{1R}^2$ and in the negative x direction when $\lambda_{1I}^2 < 3\lambda_{1R}^2$.

3 Two-soliton interaction

The two soliton solution (2SS) can be obtained from (4, 5, 6) by putting $N = 2$, the explicit form of the ith component of 2SS being

$$q_i = \frac{2 \sum_{m=1}^{2} \left(\mathcal{C}_{n+1i}^{(m)} e^{\eta_m} + e^{\eta_1 + \eta_2 + \eta_m^* + \delta_m^{(i)}} \right)}{\text{Det}[C]} \quad (10)$$

where,

$$e^{\delta_m^{(i)}} = \sum_{r,s=1}^{2} \mathcal{C}_{n+1i}^{(r)} \frac{\kappa_{ms}}{\lambda_{ms}} \left(\frac{1}{\lambda_{mr}} - \frac{1}{\lambda_{ms}} \right). \quad (11)$$

In (11) $\lambda_{ij} = \lambda_i - \lambda_j^*$ and $\eta_m = -2i\lambda_m^* t - 8i\varepsilon\lambda_m^{*3}x + i\frac{\pi}{2}$. $\text{Det}[C]$ is the determinant of 2×2 dimensional matrix obtained from $(\mathbf{C})_{ij}$ (6). The form of $\text{Det}[C]$ in terms of the spectral parameters emerges rather lengthy but it will be useful for the asymptotic analysis of the interacting solitons and will be seen later. In terms of the spectral parameters $\text{Det}[C]$ is given as

$$\text{Det}[C] = 1 + \sum_{r,s=1}^{2} e^{\eta_r + \eta_s^* + R_{rs}} + e^{2\eta_{1R} + 2\eta_{2R} + \mathcal{Q}} \quad (12)$$
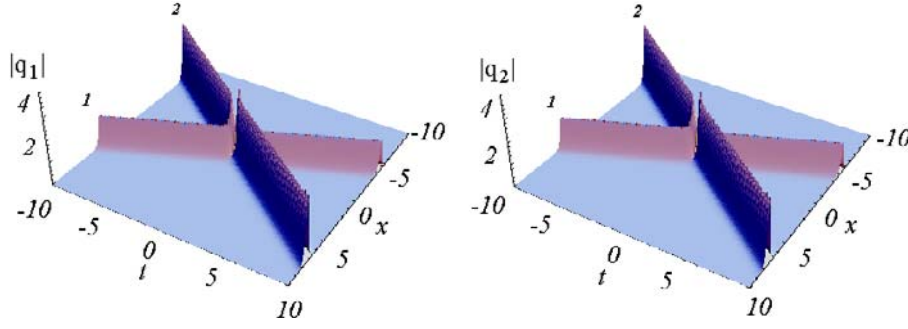
Fig. 1. Plot of $|q_1|$ and $|q_2|$ showing elastic collision.

with

$$e^{R_{rs}} = -\frac{\kappa_{rs}}{\lambda_{rs}^2}, \qquad (13)$$

$$\kappa_{rs} = \sum_{p=1}^{n} \frac{\mathcal{C}_{n+1p}^{(s)} \mathcal{C}_{n+1p}^{\star(r)}}{\alpha'_{(n+1)(n+1)}(\lambda_r^\star) \alpha'^{\star}_{(n+1)(n+1)}(\lambda_s)} \qquad (14)$$

and $e^{\mathcal{Q}} = \mathrm{Det}[\kappa]\mathrm{Det}[\mathcal{L}]$, $[\kappa] = (\frac{\kappa_{ij}}{\lambda_{ij}})$ (no sum over i, j), $[\mathcal{L}] = (\lambda_{ij})^{-1}$, $[\kappa]$ and $[\mathcal{L}]$ being 2×2 matrices. To understand the interaction in a more explicit manner we analyse the asymptotic limits of the 2SS (10). The asymptotic limit may be obtained by observing the 2SS when both the solitons are infinitely apart. This may be achieved by taking the limit $\eta_{2R} \to \pm\infty$. As a consequence ith component of the remaining soliton acquires the form

$$q_i^{(l\pm)} = \frac{\lambda_{lI} A_i^{(l\pm)} e^{i\eta_{lI}}}{\cosh(\eta_{lR} + \Phi^{l\pm})} \qquad (15)$$

where $A_i^{(l\pm)}$ and $\Phi^{(l\pm)}$ may be interpreted as the amplitudes and phases of ith component of lth soliton respectively. Notice that in general $A_i^{(l-)}$ are different from $A_i^{(l+)}$. This may occur due to exchange of energy between the solitons. The amplitude $A_i^{(l-)}$ in terms of $\mathcal{C}_{n+1i}^{(l)}$ is given as

$$A_i^{(l-)} = \frac{\mathcal{C}_{n+1i}^{(l)}}{\sqrt{\kappa_{ll}}} \cdot \qquad (16)$$

The expression for $A_i^{(l+)}$ however, is more involved and may be written in terms of $A_i^{(l-)}$ as

$$A_i^{(l+)} = A_i^{(l-)} T_i^l. \qquad (17)$$

T_i^l may be interpreted as the transition matrix and is defined as

$$T_i^1 = \frac{\left(1 - \frac{\mathcal{C}_{n+1i}^{(2)}}{\mathcal{C}_{n+1i}^{(1)}} \Lambda_2\right)}{\sqrt{1 - \Lambda_1 \Lambda_2}} \sqrt{\left(\frac{\lambda_2^\star - \lambda_1^\star}{\lambda_2 - \lambda_1}\right)\left(\frac{\lambda_2^\star - \lambda_1}{\lambda_2 - \lambda_1^\star}\right)} \qquad (18)$$

and

$$T_i^2 = \frac{\left(1 - \frac{\mathcal{C}_{n+1i}^{(1)}}{\mathcal{C}_{n+1i}^{(2)}} \Lambda_2\right)}{\sqrt{1 - \Lambda_1 \Lambda_2}} \sqrt{\left(\frac{\lambda_2^\star - \lambda_1^\star}{\lambda_2 - \lambda_1}\right)\left(\frac{\lambda_2 - \lambda_1^\star}{\lambda_2^\star - \lambda_1}\right)} \qquad (19)$$

where $\Lambda_1 = \frac{\kappa_{12}\lambda_{11}}{\kappa_{11}\lambda_{12}}$, $\Lambda_2 = \frac{\kappa_{21}\lambda_{22}}{\kappa_{22}\lambda_{21}}$. If $|T_i^l| = 1$, the solitons pass through each other without being affected in their shapes and sizes. Otherwise, we will see that the solitons exchange energy at the time of interaction. The phase shift $\Phi^{(l)}$ as a result of collision may be obtained from the following relation.

$$\Phi^{(l)} = \Phi^{(l+)} - \Phi^{(l-)} \qquad (20)$$

where $\Phi^{(l-)} = \frac{1}{2}(R_{11})$ and $\Phi^{(l+)} = \frac{1}{2}(\mathcal{Q} - R_{22})$ and consequently the phase shift becomes $\Phi^{(l)} = \frac{1}{2}(\mathcal{Q} - R_{11} - R_{22})$. It is now interesting to note that for equal values of $\mathcal{C}_{n+1i}^{(l)}$ for each component of the solitons, $|T_i^l|$ becomes unity and consequently there is no exchange of energy among the components of each soliton. However, if the relative phases among $\mathcal{C}_{n+1i}^{(l)}$ are introduced the energy of each individual component no longer remains constant due to collisions and in that case $|T_i^l| \neq 1$. This analysis has been presented graphically by plotting two soliton interaction for a two component system with the $\varepsilon = 0.1$, $\lambda_1 = -1.5 - i$ and $\lambda_1 = -0.5 - 2i$. One of the solitons is moving with group velocity -1.74 and the other is moving with a group velocity 3.07. In Figures 1 two soliton interaction is plotted with $\mathcal{C}_{n+1i}^{(l)} = 1$ for each component, where the energy profile of each component of the solitons remains unchanged. In Figures 2 two soliton interaction is plotted with non-trivial relative phases, $\mathcal{C}_{n+1\ 1}^{(1)} = \mathcal{C}_{n+1\ 2}^{(1)} = \mathcal{C}_{n+1\ 2}^{(2)} = 1$ and $\mathcal{C}_{n+1\ 1}^{(2)} = 46(1 - i)$ showing a considerable exchange of energy among the components. Figure 3 shows that the overall intensity profile, represented by $|q|$ of the 2SS remains unchanged although there are appreciable energy exchange among the components of a soliton. It is an interesting result in the context of all optical computations [2] leading to the construction of the logical binary gates.
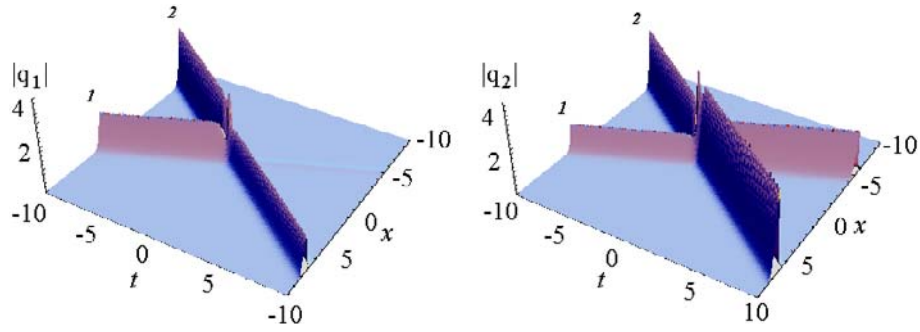
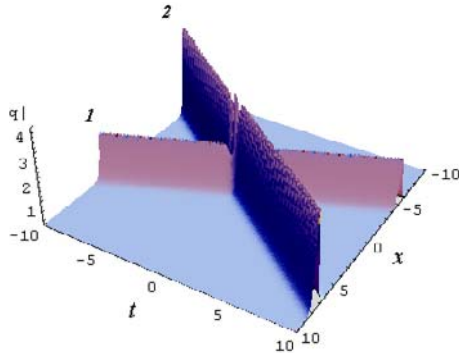**Fig. 2.** Plot of $|q_1|$ and $|q_2|$ showing inelastic collision.

Fig. 3. Plot of $|q|$ showing constant energy profile.

Fig. 4. Contour plot of three soliton interaction.

4 Three-soliton interactions

The explicit form of the three soliton solution (3SS) can be obtained from (4, 5, 6) by putting $N = 3$, the explicit form of the ith component of 3SS being

$$q_p = \frac{-2}{\text{Det}[C]}\left[\sum_{i=1}^{3} \mathcal{C}_{n+1p}^{(i)} e^{\eta_i} \right.$$
$$+ \sum_{i,j,m=1_{i\neq j\neq m}}^{3} (adj\tilde{\kappa}^j)_{mm}(adj\tilde{\mathcal{L}}^j)_{mm} e^{2\eta_{iR}+\eta_j}$$
$$\left. + \sum_{m=1}^{3} \text{Det}[\tilde{\kappa}^m]\text{Det}[\tilde{\mathcal{L}}^m] e^{\sum_{i,m=1 \atop i\neq m}^{3} 2\eta_{iR}+\eta_m} \right]$$

where,

$$\text{Det}[C] = -1 - \sum_{i,j=1}^{3} e^{\eta_i+\eta_j^*} e^{R_{ij}}$$
$$- \sum_{i=1}^{3}\sum_{k,j=1 \atop i\neq j\neq k}^{3} (adj\kappa)_{jk}(adj\mathcal{L})_{jk} e^{2\eta_{iR}+\eta_k+\eta_j^\star}$$
$$- \sum_{i,j,k=1 \atop i\neq j\neq k}^{3} (adj\kappa)_{kk}(adj\mathcal{L})_{kk} e^{2\eta_{iR}+2\eta_{jR}}$$
$$+ \text{Det}[\kappa]\text{Det}[\mathcal{L}] e^{2\eta_{1R}+2\eta_{2R}+2\eta_{3R}}$$

with $\mathcal{L}$ is a 3×3 dimensional matrix with elements (λ_{ij}^{-1}) and $[\kappa]$ is also a 3×3 dimensional matrix with elements
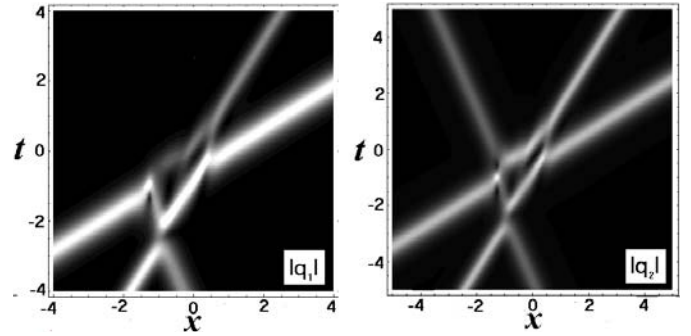
$\frac{\kappa_{ij}}{\lambda_{ij}}$ (no sum over i, j). $[\tilde{\kappa}^j]$ is the matrix $[\kappa]$, where the jth row has been replaced by $\sum_{i=1}^{3} e_{ji}\mathcal{C}_{n+1p}^{(i)}$ and $\tilde{\mathcal{L}}^j$ is the matrix where the jth row has been replaced by a unit row vector, $\sum_{i=1}^{3} e_{ji}$. The phenomena of shape changing due to exchange of energy among the components of solitons becomes more interesting in the three-soliton interaction case. It is observed that the interactions take place at three different space time points with two solitons interact at each time. Thus the energy exchange among the components in three solitons interaction may occur at three different points. This is demonstrated in Figure 4 by the contour plot of the three solitons interaction with $\epsilon = 0.1, \lambda_1 = -1.5 - i, \lambda_2 = -0.01 + 2i, \lambda_3 = -0.9 + 2i, \mathcal{C}_{n+1\ 1}^{(1)} = \mathcal{C}_{n+1\ 1}^{(2)} = \mathcal{C}_{n+1\ 1}^{(3)} = 10^{-3}, \mathcal{C}_{n+1\ 2}^{(2)} = \mathcal{C}_{n+1\ 2}^{(3)} = 1$ and $\mathcal{C}_{n+1\ 2}^{(1)} = 10^3 - 0.8i$. Two of the solitons are moving in the positive x direction with group velocities 2.5 and 6.36 and the third soliton is moving in the negative x direction with group velocity -1.74. The nature of three soliton interactions opens up another possibility of three solitons interaction where second and third solitons interact first instead of first and second solitons leaving the final configuration same in both the cases.

5 Conclusion

In conclusion, we have shown the novel shape changing phenomena associated with two soliton interaction for n component CHNLS equation. The three solitons interaction also exhibits energy exchange, but the exchange may

occur at three points. This may lead to a more flexibility in constructing logical gates in all optical computing systems. The three solitons interaction also demonstrates an interesting feature conforming the exact integrability of the system *a la* Zamolodchikov [7] and indicates the existence of Yang Baxter like relation, which will be published elsewhere.

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